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Nullity of b-Bridge Coalescence Graphs

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Abstract

The nullity η(G) (degree of singularity) of a graph G is the algebraic multiplicity of the number zero in the spectrum of G. If G is a graph containing a vertex of degree one and H be the subgraph obtained from G, by deleting this vertex together with the vertex adjacent to it then, η(G) = η(H). In this paper, we proved that nullity of a graph is the maximum number of independent variables in a high zero-sum weighting for it. The above procedures are applied to evaluate the nullity of b-bridge coalescence graphs. They are also applied to determine the nullity of edge introducing between t-tuple coalescence graphs and nullity of paths introducing between (n, m)-tuples of coalescence graphs.

Introduction

A graph G is said to be a singular graph provided that its adjacency matrix A(G) is a singular matrix. The algebraic multiplicity of the number zero in the spectrum of the graph G is called a nullity (degree of singularity), and is denoted by η(G). See [5] or [6].

A vertex weighting of a graph G is a function f: V(G) →R where R is the set of real numbers, which assigns a real number (weight) to each vertex. The weighting of G is said to be non-trivial if there is at least one vertex v for which $f(v) \neq 0$. A non-trivial vertex weighting of a graph G is called a zero-sum weighting provided that for each $v \in V(G)$, $\sum w(u)=0$, where the summation is taken over all u in the neighborhood of v, $u \in N_G(v)$. A graph G is singular, if and only if there is a non-trivial zero-sum weighting for G.

Out of all zero-sum weightings of a graph G, a high zero-sum weighting of G is one that uses maximum number of non-zero independent variables.

In Fig.1.1, the weighting for the graph G is a high zero-sum weighting that uses three independent variables, hence η(G)=3

It is clear that, the complete graph K_n, the even path P_{2n} and the cube Q_3 has no non-trivial zero-sum weighting. While the star graph $S_{1,n-1}$,has a high zero-sum weighting that uses n-2 independent variables, $\eta(S_{1,n-1})=n-2$ and for the cycle C_{4n}, uses 2.

We start this section with the following well known results.

Corollary 1.1: [4] and [7, p.234] (End Vertex Corollary)(E.V.C.)

If G is a bipartite graph with an end vertex, and if H is an induced subgraph of G obtained by deleting this vertex together with the vertex adjacent to it, then η(G)=η(H).

Proposition 1.2:[8] Let v be any vertex (which need not be a cut-vertex) of a graph G with order at least 2. Then, $\eta(G) -1 \leq \eta(G-v) \leq \eta(G) +1$.

This paper, is based on M.Sc. Thesis submitted to the Faculty of Science, University of Zakho by the second author under the supervision of the first.

Theorem 1.3:[8] Let v be a cut-vertex of a graph G of order n and $G_1, G_2,..., G_s$ be all components of G-v. If there exists a component, say G_1, among $G_1, G_2,..., G_s$ such that $\eta(G_1)=\eta(G_1+v)+1$, then $\eta(G)=\eta(G-v) - 1=\sum_{i=1}^{s}\eta(G_i) - 1$.

Theorem 1.4:[8] Let v be a cut-vertex of a graph G of order n and G_1 be a component of G-v. If $\eta(G_1)=\eta(G_1 + v)-1$, then $\eta(G)=\eta(G_1)+\eta(G-G_1)$

The **vertex identification graph** $G_1 \bullet G_2$ is obtained from G_1 and G_2 by identifying the vertex $u \in G_1$ with $v \in G_2$. The **edge introduced graph** G_1**:**G_2 of two graphs G_1 and G_2 is obtained by introducing the edge e=uv between a vertex $u \in G_1$ with a vertex $v \in G_2$

Two vertices of a graph G are said to be of the **same type (coneighbors)** if they are not adjacent and have the same neighbors. Thus, two vertices v_i, v_j of the same type have the same row vectors $R_i=R_j$ describing them, where R_i and R_j are the i^{th} and j^{th} row vectors of A, corresponding to the vertices v_i and v_j, i, j= 1, 2, …, p. Each pair of such (same type) vertices results in two dependent rows which yield a zero in spectra of the graph G. It is clear that the occurrence of m equal rows contributes (m-1) to the nullity. Two adjacent vertices with the same neighbor are call **semi- coneighbors** (S.C.). Moreover any pair of semi-coneighber vertices must have the same weight in any zero sum weighting of the graph.

Lemma1.5: (Coneighbor Lemma) (C.L.) If a graph G contains a pair u and v of coneighber vertices, then $\eta(G)= \eta(G-v)+1$.

Proof: In the adjacency matrix A(G), the corresponding rows to u and v will be identical. ■

The graph Γ_3 of order 9 possesses a high zero-sum weighting that uses exactly 3 independent variables namely x, y and z thus $\eta(\Gamma_3)=3$, and the vector $(x, -x, x, -x, y, -y-z, z, -y-z, z)^T$ is in the null space of $A(\Gamma_3)$.

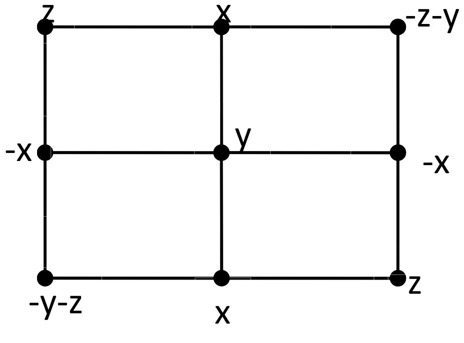


Figure -1: The graph Γ_3.

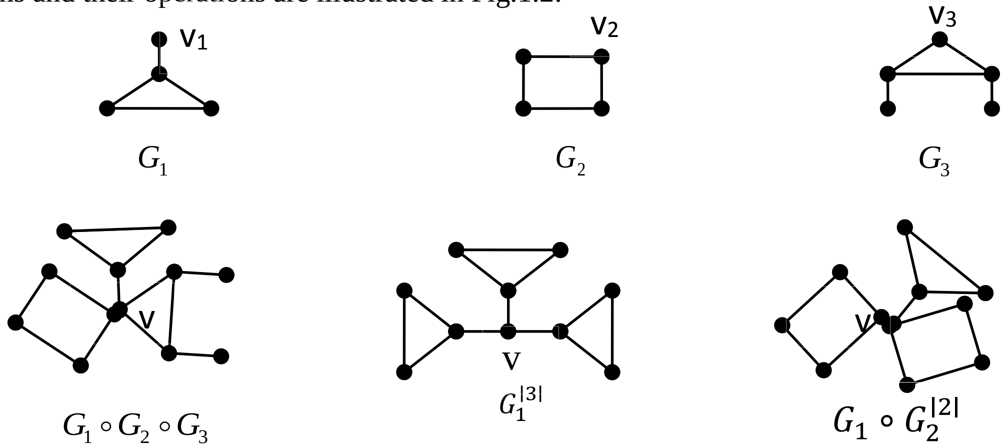
To **identify** non adjacent vertices u and v of a graph G is to replace the two vertices by a single vertex adjacent to all vertices that are adjacent to either u or v in G, we shall denote the resulting graph by G/{u, v}.

Definition 1.6: [1] and [2, p.31] Let (G_1, u) and (G_2, v) be two disjoint graphs rooted at vertices u and v, respectively. We attach G_1 to G_2 (or G_2 to G_1) by identifying the vertex u of G_1 with the vertex v of G_2. Vertices u and v are called **vertices of attachment**. The vertex formed by their identification is called the **coalescence vertex**. The resulting graph $G_1 \circ G_2$ is called the **coalescence (vertex identification)** of G_1 and G_2.

Definition 1.7: [2, p.31] Let $\{(G_1, v_1), (G_2, v_2), \ldots, (G_t, v_t)\}$ be a family of not necessary distinct connected graphs with roots $v_1, v_2, \ldots, v_t$, respectively. A connected graph $G= G_1 \circ G_2 \circ \ldots \circ G_t$ is called the **multiple coalescence** of $G_1, G_2,\ldots,G_t$ provided that the vertices $v_1, v_2, \ldots, v_t$ are identified to reform the coalescence vertex v.

The **t-tuple coalescence graph** is denoted by $G^{|t|}$ is the multiple coalescence of t isomorphic copies of a graph G. In the same way, we shall use $G_1 \circ G_2^{|t|}$ to denote the multiple coalescence of G_1 and t copies of G_2. Therefore, all coalescence graphs have v as a common cut-vertex.

Some graphs and their operations are illustrated in Fig.1.2.



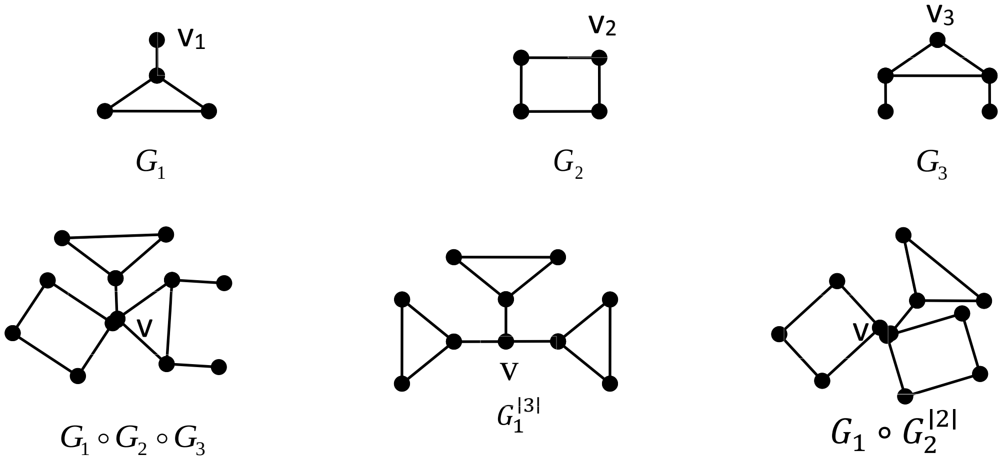


Figure -2: Multiple coalescence $G_1 \circ G_2 \circ G_3$, 3-tuple coalescence $G_1^{|3|}$ and coalescence of both $G_1 \circ G_2^{|2|}$.

Definition 1.8: [2] and [10] Let G be a graph consisting of n vertices and L = {H_1, H_2, …, H_n} be a family of rooted graphs. Then, the graph formed by identifying the root of H_k to the kth($1 \leq k \leq n$) vertex of G is called the **generalized coalescence graph** and is denoted by G(L). G itself is called the **core** of G(L). If each member of L is isomorphic to the rooted graph H, then the graph G(L) is denoted by G(H). Recall G_1, G_2 and G_3 from Fig. 1.2 then, we have

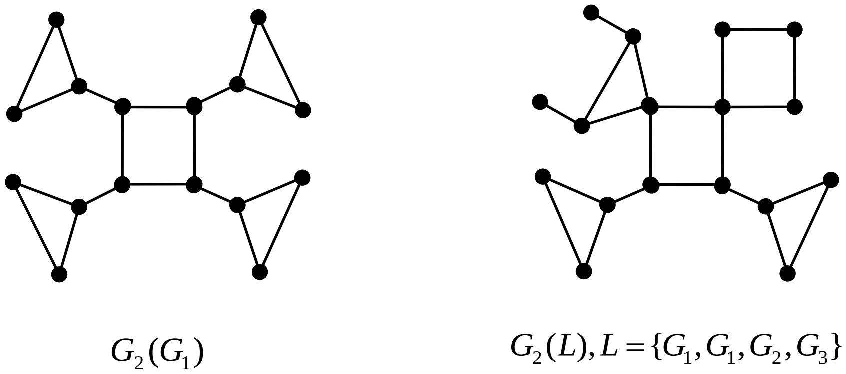


Figure -3.: Generalized coalescence graphs.

2 b-Bridge Tuple Graphs

In this section, we introduce new types of coalescence graphs and evaluate their nullities.

Definition 2.1: Let (G_i,u_i) be b rooted graphs, i=1,2,…,b, where u_i is a root of G_i. Let K_1 be the trivial graph with vertex v, then the **b-bridge tuple graph**$K_1^{b(G_i)}$is a graph obtained from $K_1 \cup_{i=1}^{b} G_i$with extra b bridges, namely vu_i, i=1,2,…,b. See Fig. 4 where $G_i=C_4$, for each i.

Next, we shall give the following results

Proposition 2.2: Let, the root u_{1i} be an end vertex of each path P_{2n_i+1},P_{2n_i}, for $1 \leq i \leq b$, $b \geq 2$, then, i) $\eta(K_1^{b(P_{2n_i+1})})$=b-1 and ii) $\eta(K_1^{b(P_{2n_i})})$=1.

Proof: i)This graph is a type of a star like graph, the zero-sum weighting for such a graph is easily determined using exactly b-1 independent variables. $\eta(K_1^{b(P_{2n_i+1})})=\eta(\cup_{i=1}^{b-1} P_{2n_i+1})$=b-1, by [9, Lemma1.3.9].

ii) Applying E.V.C. ($\sum_{i=1}^{n} n_i$) times, we get $\eta(K_1^{b(P_{2n_i})})=\eta(K_1)=1$. ■

Proposition 2.3: If $(G_i,u_i)=(C_3,u_{1i})$, for any vertex u_{1i} of C_3,$1 \leq i \leq b$, $b >1$, then $\eta(K_1^{b(C_3)})=0$.

Proof: By using weighting procedure with the fact that semi-coneighbor vertices have the same weight, we get: $2x_1=2x_2=\ldots=2x_b$ and $x_1+x_2+\ldots+x_b=0$, which implies $x_i=0$ for i=1, 2, …, b, then, there exist no non-trivial zero-sum weighting for the graph $K_1^{b(C_3)}$, hence $\eta(K_1^{b(C_3)})=0$.

Proposition 2.4: If $(G_i,u_i)= (C_4,u_{1i})$ for any vertex u_{1i} of C_4,$1 \leq i \leq b$, then $\eta(K_1^{b(C_4)})$=2b-1.

Proof: There exist b pairs of coneighbor vertices, hence applying C.L. b times, that is removing a vertex from each of such a pair, we get: $\eta(K_1^{bC_4})=b+\eta(K_1^{bP_3})$

Hence, by Prop. 2.2, $\eta(K_1^{bC_4})$=b+(b-1)=2b-1. See Fig. 4.Also, by Theorem 1.3, we get the result.

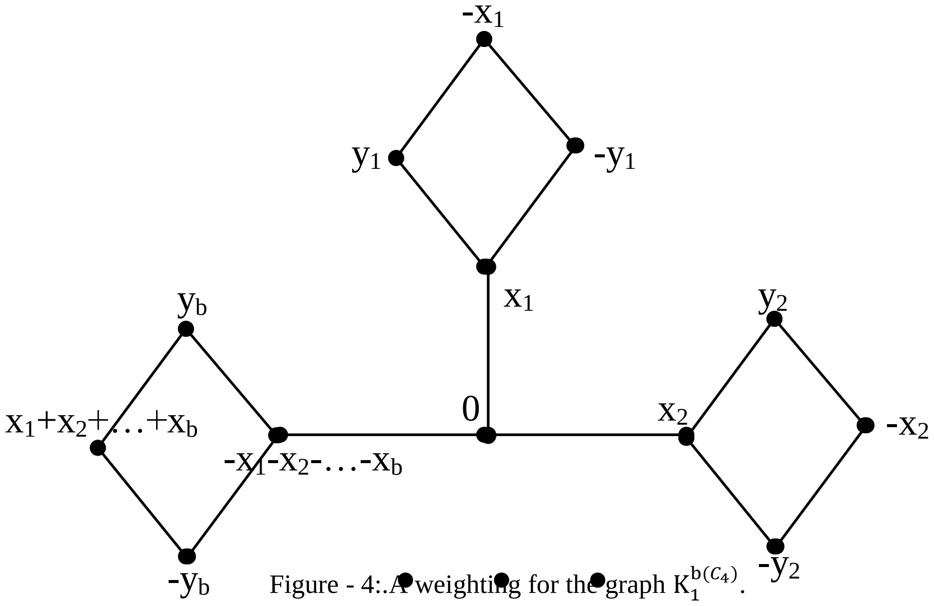


Figure - 4:.A weighting for the graph $K_1^{b(C_4)}$.

Morever, applying the same weighting procedure, it is clear that for each i ≥1,$\eta(K_1^{b(C_{4i})})$=2b-1. ■

Proposition 2.5: The nullity of the graph $K_1{}^{b(C_{4i+j})}$ is given by, $\eta(K_1^{b(C_{4i+j})})=\begin{cases} 0, \ if j = 1 \ or \ 3, \\ 1, \ if j = 2. \end{cases}$

Where i≥1, such a result can be explained by the following simple equations.

Proof: We use weighting procedure as we have given in proving Prop. 2.3 and Prop. 2.4. If j=1,we get: $-2x_1=-2x_2=\ldots=-2x_b$ and $x_1+x_2+\ldots+x_b=0$.

Thus, we have $x_i=0$, then there exist no non-trivial zero-sum weighting, gives $\eta(K_1^{b(C_{4i+1})})=0$.

If j=3, also we get: $2x_1=2x_2=\ldots=2x_b$ and $x_1+x_2+..+x_b=0$, we get $x_i=0$, then, there exist no non-trivial zero-sum weighting, thus ($K_1^{b(C_{4i+3})}$)=0..

If j=2, we get: $-2y_1=-2y_2=\ldots=-2y_b \Rightarrow y_1=y_2=\ldots=y_b$, then $\eta(K_1^{b(C_{4i+2})})=1$.

Also, we can apply Theorem 1.4 to get the result. ■

Proposition 2.6: The nullity of the graph$K_1^{b(K_{n,n})}$is $\eta(K_1^{b(K_{n,n})})$=2b(n-1)-1, n> 1.

Proof: Applying C.L. to all coneighbor vertices, we get:$\eta(K_1^{b(K_{n,n})})$=b(n-1+n-2)+$\eta(K_1^{b(P_3)})$,

Then, by Prop.2.2(i), we get :$\eta(K_1^{b(K_{n,n})})$=b(2n-3)+b-1=2b(n-1)-1. ■

Proposition 2.7: If $(G_i,u_i)= (K_n,u_{1i})$ for any vertex u_{1i} of K_n, $1 \le i \le b$, then $\eta(K_1^{b(K_n)})=0$, n≥3.

Proof: Starting with the first component and apply semi-coneighbor lemma, then all vertices which are not adjacent with v must have the same weight, say x and each vertex adjacent with v must be weighted by -(n-2)x, while v must be weighted by -(n-1)x. If we add the weights over the neighborhood of v, we get b[-(n-2)x]=0, therefore, (n-2)x=0 if n≠2 then x=0, and there exist no non-trivial zero-sum weighting for$K_1^{b(K_n)}$, therefore, $\eta(K_1^{b(K_n)})=0$.

If n=2, then by Prop. 2.3 $\eta(K_1^{b(K_2)})=1$. Hence the proof is complete. ■

Proposition 2.8: If $(G_i,u_i)=(Q_3^{(2)},u_{1i})$,$1 \le i \le b$, b>1, then $\eta(K_1^{b(Q_3^{(2)})})=1$.

Proof : Applying Theorem 1.4., we get:$\eta(K_1^{b(Q_3^{(2)})})=\sum_{i=1}^{b-1} \eta(Q_3^{(2)})+ \eta(Q_3^{(2)}+v)$, applying E.V.C:

$\eta(K_1^{b(Q_3^{(2)})}) =\sum_{i=1}^{b-1} \eta(Q_3^{(2)})+ \eta(Q_3^{(2)}-v_{11})$, then by [9, Prop. 2.5.1 and Lemma 2.5.2], we obtain, $\eta(K_1^{b(Q_3^{(2)})})=1$.■

Now we generalize the above proposition to

Proposition 2.9: For n≥1, the nullity of the graph $K_1{}^{b(Q_{2n}^{(2)})}$ is $\eta(K_1{}^{b(Q_{2n}^{(2)})})=b\binom{n}{n/2}-1$.

Proof: Applying Theorem 1.3, once : $\eta(K_1{}^{b(Q_{2n}^{(2)})})=\sum_{i=1}^{b} \eta(Q_{2n}^{(2)}) - 1$, then by [9, Prop. 2.5.1], we have η($K_1{}^{b(Q_{2n}^{(2)})}$)=b$\binom{n}{n/2} - 1$.■

Problem: For more generalizations of the above result one can ask for η($K_1{}^{b(Q_{n_i})}$), where, $1 \leq i \leq b$, b>1.

3 b-Bridge Tuple Cycles

If the core vertex v of K_1 in Definition 2.1 is replaced by a b-cycle C_b, with V(C_b)={v_1,v_2,…,v_b}, then we call such a b-bridge tuple graph by a **b-bridge tuple cycle** and symbolized it by $C_b^{b(G_i)}$. That is $C_b^{b(G_i)}$ is obtained from $C_b \bigcup_{i=1}^{b} G_i$ with extra edges v_iu_{1i}, u_i∈V(G_i), for i=1,2,…,b. See Fig. 5

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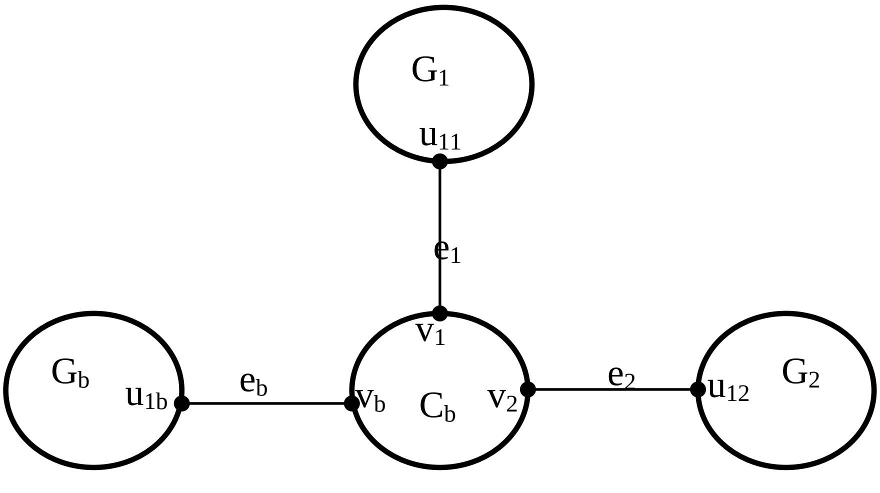

Figure -5: The b-bridge tuple cycle graph $C_b^{b(G_i)}$.

Proposition 3.1: If G_i=K_1 for all i, $1 \leq i \leq b$, then η($C_b^{b(K_1)}$)=0.

Proof : The result holds, by applying E.V.C. b-2 times, ■

The same result holds if K_1 is replaced by P_{2n_i+1}in the above proposition, in which u_i is an end vertex of P_{2n_i+1}, for i=1, 2, …,b.

Proposition 3.2: If $G_i \cong P_{2n_i}$,then η($C_b^{b(P_{2n_i})}$)= $\begin{cases} 2\,, & if b \equiv 0 (mod 4), \\ 0, & otherwise. \end{cases}$

Proof : Applying E.V.C. nb times we get the result.■

Proposition 3.3: If G_i=K_p, p >2 and b=3,4,5, then η($C_b^{b(K_p)}$)=0.

Proof : If p=1, it is clear by Prop. 3.1, that $C_b^{b(K_p)}$ is non-singular, while if p=2 then by Prop.3.2 we get: η($C_b^{b(K_2)}$)= $\begin{cases} 2, & if b \equiv 0 (mod 4), \\ 0, & otherwise. \end{cases}$

Assume that p > 2, the weighting for $C_b^{b(K_p)}$ can be constructed, but the coefficients of the independent variables are rational numbers, then there exist no non-trivial zero-sum weighting for $C_4^{b(K_p)}$.

For b=5, using row elimination method, we can prove that the determinant of the coefficients of the above system is not equal to zero. Hence, there exists no non-trivial zero-sum weighting for$C_5^{b(K_p)}$, which means that it is non-singular.■

Open problem: Evaluate $\eta(C_b^{b(K_p)})$, where b ≥6.

Proposition 3.4: If $G_i=C_n$, then $\eta(C_b^{b(C_n)})=\begin{cases} b, & if n \equiv 0(mod 4), \\ 0, & otherwise. \end{cases}$

Proof : If n=3, the prove follows by Prop. 3.3.If n=4, it is clear that $\eta(C_b^{b(C_4)})$=b.

Since, there exist b pairs of coneighbor vertices, by removing a vertex out of each such a pair of vertices, we get $\eta(C_b^{b(C_n)})=b+\eta(C_b^{b(P_3)})$ =b+0=b. ■

Proposition 3.5: The nullity of the graph $C_b^{b(P_{2n+1}\bullet C_{4m})}$ is given by$\eta(C_b^{b(P_{2n+1}\bullet C_{4m})})=\begin{cases} b+2, & if b \equiv 0(mod\ 4), \\ b, & otherwise. \end{cases}$

Proof: Applying Theorem 1.3, b times,$\eta(C_b^{b(P_{2n+1}\bullet C_{4m})})$=2b-b+ $\eta(C_b)$ =b+$\begin{cases} 2, & if b \equiv 0(mod\ 4), \\ 0, & otherwise. \end{cases}$■

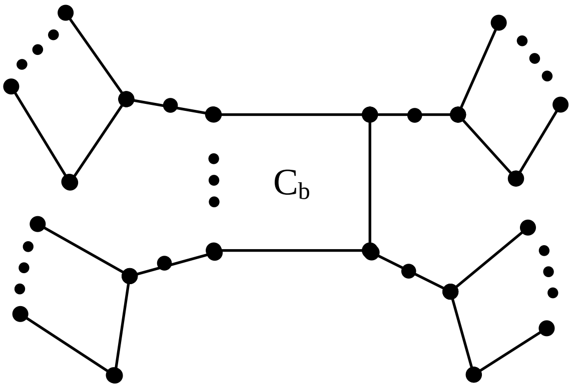

Figure - 6:The graph$C_b^{b(P_{2n+1}\bullet C_{4m})}$.

Proposition 3.6: The nullity of the graph$C_b^{b(P_{2n}\bullet C_{4m})}$ is $\eta(C_b^{b(P_{2n}\bullet C_{4m})})$=b.

Proof: Is similar to the proof of Prop. 3.5.■

4 Sequential Edge Introducing Between t-Tuple Coalescence Graphs

In this section, we determine the nullity of sequential edge introducing between t-tuple coalescence graphs, for some special graphs.

Let $\{(G_1^{|t|}, v_1), (G_2^{|t|}, v_2),\ldots, (G_r^{|t|}, v_r)\}$ be a set of r vertex disjoint connected t-tuple graphs and the graph $H(G_i^{|t_i|}:r)=G_1^{|t|}: G_2^{|t|}: \ldots : G_{r-1}^{|t|}: G_r^{|t|}$for all t_i=t, be obtained from $G_1^{|t|}, G_2^{|t|}, \ldots, G_r^{|t|}$by introducing r-1 new edges between the rooted vertices v_i v_{i+1}, i=1, 2,.., r-1 of these t-tuple coalescence graphs $G_i^{|t_i|}$, for r ≥ 2 and t> 2.

If each $G_i^{|t|}$ is $P_p^{|t|}$, then the graph, H($P_p^{|t|}$**:** r) is illustrated in Fig. 7.

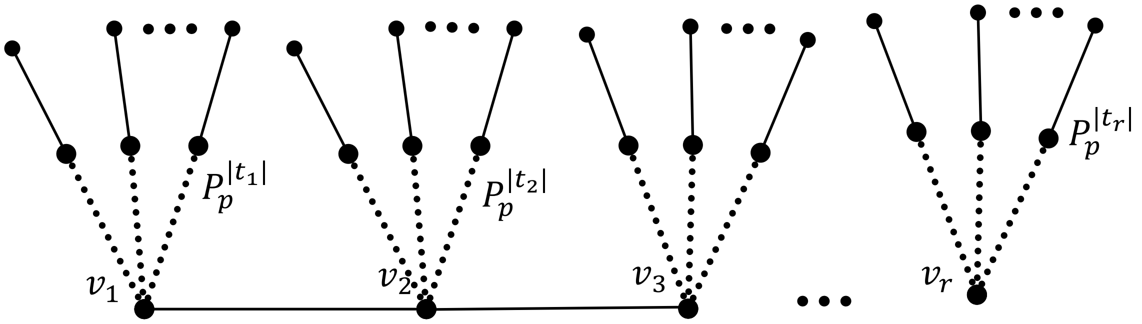

Figure -7:The graph H($P_p^{|t|}$: r).

The nullity of H($P_p^{|t|}$**:** r) is determined in the next proposition.

Proposition 4.1: The nullity of the graph H($P_p^{|t|}$**:** r) is given by

a) If p=2n, where the attachment vertex is at any vertex, then η(H($P_p^{|t|}$**:** r))=r(t-1).
b) If p=2n+1, where the attachment vertex is at a neighbor of an end vertex or at a vertex with zero weight, then η(H($P_p^{|t|}$**:** r)) =r(2t-1).
c) If p=2n+1, where the attachment vertex is at a vertex with non-zero weight, then

$$\eta(H(P_p^{|t|}: r)) = \begin{cases} 1, if r is odd, \\ 0, if r is even. \end{cases}$$

Proof: a) Applying E.V.C. n(r-1) times, we get:

η(H($P_p^{|t|}$**:** r))=t-1+t-1+…+η($P_p^{|t|}$), by [2, Prop. 3.1.2(i)] η($P_p^{|t|}$)=t-1, then we get η(H($P_p^{|t|}$**:** r))=r(t-1).

b) Applying E.V.C. r times, η(H($P_p^{|t|}$**:** r))=2t-1+2t-1+…+2t-1=r(2t-1).

For c) If r is odd, applying E.V.C. (n+(r+1)/2) times we get

η(H($P_p^{|t|}$**:** r))= η($P_p^{|t|}$), by [2, Prop. 3.1.2(iii)], η($P_p^{|t|}$)=1, therefore,

η(H($P_p^{|t|}$**:** r))=1.

If r is even, by using E.V.C.(n+(r+1)/2) times we get the result.■

Let $C_p^{|t|}$ be the t-tuple coalescence graph of a cycle graph C_p,then H($C_p^{|t|}$**:** r) have order r(t(p-1)+1) and size r(tp+1)-1, and the diam(H($C_p^{|t|}$**:** r))=2diam(C_p)+r-1.

The nullity of H($C_p^{|t|}$**:** r) is determined in the next proposition.

Proposition 4.2: The nullity of the graph H($C_p^{|t|}$**:** r) is given by

a) If p=4n, n=1, 2, …, then. $\eta(H(C_p^{|t|}\colon r))=\begin{cases} rt+1, ifrisodd, \\ rt, ifriseven. \end{cases}$

b) If p=4n+2, n=1, 2, …, then. $\eta(H(C_p^{|t|}\colon r))=r(t-1)$

c) If p is odd, then. $\eta(H(C_p^{|t|}\colon r))=0$.

Proof: If r is odd, by applying Theorem 1.4, r times we get: $\eta(H(C_p^{|t|}\colon r))=t+t+\ldots+t+\eta(P_r)=rt+1$.

If r is even, by applying Theorem 1.4, r times we get: $\eta(H(C_p^{|t|}\colon r))=t+t+\ldots+t+\eta(P_r)=rt$.

a) Again applying Theorem 1.3, r times to this case we get :

$\eta(H(C_p^{|t|}\colon r))=t-1+t-1+\ldots+t-1=rt-r=r(t-1)$.

b) The proof is easy.■

Denote the t-tuple coalescence for complete bipartite graph $K_{n,n}$by $H(K_{n,n}^{|t|})$. Then, the nullity of $H(K_{n,n}^{|t|}\colon r)$ is determined in the next proposition.

Proposition 4.3: The nullity of the graph $H(K_{n,n}^{|t|}\colon r)$ is given by

a) If n=1, then $\eta(H(K_{1,1}^{|t|}\colon r))=r(t-1)$.

b) If n=2, then $\eta(H(K_{2,2}^{|t|}\colon r))=\begin{cases} rt+1, & ifrisodd, \\ rt, & ifriseven. \end{cases}$

c) If n>2, then $\eta(H(K_{n,n}^{|t|}\colon r))=\begin{cases} rt(2n-3)+1, & ifrisodd, \\ rt(2n-3), & ifriseven. \end{cases}$

Proof: The proof follows from Prop. 4.1(a).

a) The proof follows from Prop. 4.3 (a).

b) If r is odd, removing all but one of the coneighbor vertices of $K_{n,n}$ in each$K_{n,n}^{|t|}$, we get: $\eta(H(K_{n,n}^{|t|}\colon r))=t(2n-3)+t(2n-3)+\ldots+t(2n-3)+\eta(H(P_{2n+1}^{|t|}\colon r))$, by Prop.4.1(c) $\eta(H(P_{2n+1}^{|t|}\colon r))=1$, then$\eta(H(K_{n,n}^{|t|}\colon r))=r(t(2n-3))+1$.

If r is even, then by removing all but one of the coneighbor vertices of each $K_{n,n}$ in each $K_{n,n}^{|t|}$, we get: $\eta(H(K_{n,n}^{|t|}\colon r))=t(2n-3)+t(2n-3)+\ldots+t(2n-3)+\eta(H(P_{2n+1}^{|t|}\colon r))$

By, Prop. 4.1(c)$\eta(H(P_{2n+1}^{|t|}\colon r))=0$, then $\eta(H(K_{n,n}^{|t|}\colon r))=rt(2n-3)$.■

Denote the t-tuple graphcoalescence for complete graph K_n by$H(K_n^{|t|})$. Then the nullity of $H(K_n^{|t|}\colon r)$ is determined in the next proposition.

Proposition 4.4: If n=2, then $\eta(H(K_2^{|t|}\colon r))=r(t-1)$, while, if $n \geq 3$, then $\eta(H(K_n^{|t|}\colon r))=0$.

Proof: Follows from Proposition 4.3.1.(a) and For $n \geq 3$, by [2, Prop. 3.1.3(ii)]

$\eta(H(K_n^{|t|}\colon r))=0$, also by applying S.C.L. the graph $H(K_n^{|t|}\colon r)$ is non-singular. ■

5 Path Introducing Between (n,m)-Tuples Coalescence Graphs

In this section, we are going to determine the nullity of an even (odd) path introduced between n-tuple coalescence and m-tuple coalescence graphs.

Definition 5.1: Let F be the graph obtained by an even(odd) path introduced between n-tuple coalescence G_i and m-tuple coalescence H_j, ∀i, j,i=1, 2, …, n and j=1, 2, …, m, if the coalesced vertex is at an end vertex. We denote our notation by F=$\prod_{i=1}^{n} G_i : P_p : \prod_{j=1}^{m} H_j$.

The nullity of an even (odd)path introduced between n-tuple coalescence and m-tuple coalescence of path graphs of the form P_r is studied in the next proposition.

Proposition 5.2: If each G_i and $H_j$is a path, then

i) $\eta\left(\prod_{i=1}^{n} P_{2i} : P_{2k} : \prod_{j=1}^{m} P_{2j}\right) = n + m - 2.$

ii) $\eta\left(\prod_{i=1}^{n} P_{2i} : P_{2k+1} : \prod_{j=1}^{m} P_{2j}\right) = n + m - 1.$

iii) $\eta\left(\prod_{i=1}^{n} P_{2i+1} : P_{2k} : \prod_{j=1}^{m} P_{2j+1}\right) = 0.$

iv) $\eta\left(\prod_{i=1}^{n} P_{2i+1} : P_{2k+1} : \prod_{j=1}^{m} P_{2j+1}\right) = 1.$

v) $\eta\left(\prod_{i=1}^{n} P_{2i+1} : P_{2k} : \prod_{j=1}^{m} P_{2j}\right) = m.$

vi) $\eta\left(\prod_{i=1}^{n} P_{2i+1} : P_{2k+1} : \prod_{j=1}^{m} P_{2j}\right) = m - 1.$

Proof: For all cases, applying E.V.C. we get the results. ■

The nullity of an even (odd) path introduced between n-tuple coalescence and m-tuple coalescence of cycle graphs of the form C_r is studied in the next proposition.

Proposition 5.3: If each G_i and H_j is a cycle whose order is multiple of 4, then

i) $\eta\left(\prod_{i=1}^{n} C_{4i} : P_{2k} : \prod_{j=1}^{m} C_{4j}\right) = n + m.$

ii) $\eta\left(\prod_{i=1}^{n} C_{4i} : P_{2k+1} : \prod_{j=1}^{m} C_{4j}\right) = n + m + 1.$

Proof : i) Applying Theorem 1.4 for each C_{4i} and C_{4j} we get:

$\eta\left(\prod_{i=1}^{n} C_{4i} : P_{2k} : \prod_{j=1}^{m} C_{4j}\right) = n + m + \eta(P_{2k})$, but $\eta(P_{2k}) = 0$, then,$\eta\left(\prod_{i=1}^{n} C_{4i} : P_{2k} : \prod_{j=1}^{m} C_{4j}\right) = n + m.$

ii) Applying Theorem 1.4 for each C_{4i} and C_{4j} we get:

$\eta\left(\prod_{i=1}^{n} C_{4i} : P_{2k+1} : \prod_{j=1}^{m} C_{4j}\right) = n + m + \eta(P_{2k+1})$, but $\eta(P_{2k+1}) = 1$, then

$\eta\left(\prod_{i=1}^{n} C_{4i} : P_{2k} : \prod_{j=1}^{m} C_{4j}\right) = n + m + 1.$■

The nullity of an even(odd)path introduced between n-tuple coalescence and m-tuple coalescence of complete bipartite graphs of the form $K_{r,r}$ is studied in the next proposition.

Proposition 5.4: If each of G_i and $H_j$is $K_{r,r}$, r≥2, then

i) $\eta\left(\prod_{i=1}^{n} K_{r,r} : P_{2k} : \prod_{j=1}^{m} K_{r,r}\right) = (2r - 3)n + m.$

ii) $\eta\left(\prod_{i=1}^{n} K_{r,r} : P_{2k+1} : \prod_{j=1}^{m} K_{r,r}\right) = (2r - 3)n + m + 1.$

Proof : i) By applying C.L. and using Prop.5.3 we get the result.

ii) By applying C.L. and using Prop. 5.3 we get the result. ■

The nullity of an even (odd) path introduced between n-tuple coalescence and m-tuple coalescence of complete graphs is studied in the next proposition.

Proposition 5.5: If each G_i and $H_j$is a complete graph, then

i) $\eta\left(\prod_{i=1}^{n} K_{ri} : P_{2k} : \prod_{j=1}^{m} K_{rj}\right) = n + m - 2, if r_i, r_j = 2$, for all i, j.
ii) $\eta\left(\prod_{i=1}^{n} K_{ri} : P_{2k+1} : \prod_{j=1}^{m} K_{rj}\right) = n + m - 1, if r_i, r_j = 2$, for all i, j.
iii) $\eta\left(\prod_{i=1}^{n} K_{ri} : P_{2k} : \prod_{j=1}^{m} K_{rj}\right) = 0, if r_i, r_j \geq 3$, for all i, j
iv) $\eta\left(\prod_{i=1}^{n} K_{ri} : P_{2k+1} : \prod_{j=1}^{m} K_{rj}\right) = 0, if r_i, r_j \geq 3$, for all i, j.

Proof: Follows from Prop. 5.2(i).

(i) Follows from Prop, 5.2(ii).
(ii) For n ≥ 3, then by [2, Prop. 3.1.3(ii)], η($K_n^{|t|}$)=0 and applying S.C.L., then, $\prod_{i=1}^{n} K_{ri}$ and $\prod_{j=1}^{m} K_{ri}$are non-singular, therefore,$\eta\left(\prod_{i=1}^{n} K_{ri} : P_{2k} : \prod_{j=1}^{m} K_{rj}\right) = 0$.
(iii) Is similar to the above case. ■

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